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An N-mode squeezed vacuum state in Fock space as an entangled state

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Abstract

Using the technique of integration within the ordered product (IWOP) of operators, we show that the operator $U = \exp\left[\mathrm{i}r\left(\sum_{i=1}^{n-1} Q_i P_{i+1} + Q_n P_1\right)\right]$ is an N-mode squeezing operator for the N-mode quadratures exhibiting the standard squeezing. The corresponding squeezed vacuum state in N-mode Fock space is derived, and the entanglement involved in such a state is also explained. We present an optical network for producing the N-mode squeezed state.

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1. Introduction

The entangled states and the entanglement have been important topic since 1970s due to their wide applications in optical communication, quantum teleportation and quantum state engineering [1–6]. Many efforts have been made to find new entangled states and a new form of squeezing operators so that new experimental implementation could be proposed [7–9]. In [10], the two-mode squeezed state, which is composed by the idler mode and signal mode resulting from a parametric down conversion amplifier, is a typical entangled state of continuous variable. Theoretically, it is constructed by the two-mode squeezing operator S acting on the vacuum state $|00\rangle$, i.e., $S|00\rangle = \sec h\lambda \exp\left(-a_1^\dagger a_2^\dagger \tanh\lambda\right)|00\rangle$, where λ is a squeezing parameter. Using the relation between the Bose operators $\left(a_i, a_i^\dagger\right)$ and the coordinate, momentum operators $Q_i = \frac{1}{\sqrt{2}}\left(a_i + a_i^\dagger\right)$, $P_i = \frac{1}{\mathrm{i}\sqrt{2}}\left(a_i - a_i^\dagger\right)$, Hongyi Fan found a new operator $S = \exp[\mathrm{i}\lambda(Q_1P_2 + Q_2P_1)]$, which actually squeezes the entangled state $|\eta\rangle$ [11, 12]. In [13], he extended his idea to three-mode and proved $U = \exp\left[\mathrm{i}r\left(Q_1P_2 + Q_2P_3 + Q_3P_1\right)\right]$ is also a squeezing operator in the three-mode Fock space, and its corresponding squeezed vacuum state is also an entangled state. An interesting problem thus naturally arises. Can Fan' s idea be extended to the N-mode case? That is to say, is the unitary operator

$U = \exp\left[\mathrm{i}r\left(\sum_{i=1}^{n-1} Q_i P_{i+1} + Q_n P_1\right)\right]$ also a squeezing operator in the N-mode Fock space? If yes, then what is its corresponding squeezed vacuum state? Is it also an entangled state?

The N-mode squeezing operator is a larger symmetry algebra of the Virasoro algebra. To answer these questions we must first derive the normal product form of U and then analyze if the squeezing exists, and how behaves the state $U|\mathbf{0}\rangle$ (where $|\mathbf{0}\rangle$ stands for the N-mode vacuum state). The paper is organized as follows. In section 2, we use the IWOP technique to expand normally ordered U. In sections 3–4 we examine the properties of the state $U|\mathbf{0}\rangle$, and find that it just makes the variances of the N-mode quadrature operators behave as that of the two-mode case. In section 5 we discuss how to design an optical network to realize the new N-mode squeezed vacuum state.

2. Normal product form of U

Because operators $Q_1P_2, Q_2P_1, \ldots, Q_{n-1}P_n$ and Q_nP_1 neither commute with each other nor make any close relation with the Lie algebra by themselves, it seems difficult to disentangle U. Thus we must appeal to the IWOP technique. We rewrite U as

$$U = \exp[\mathrm{i}r Q_i A_{ij} P_j],$$
$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}, \qquad Q_i = (Q_1, Q_2, \ldots Q_n),\ P_j = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix}, \tag{1}$$

where the repeated indices represent the Einstein summation notation and A is an $n \times n$ matrix. Using the Baker–Hausdorff formula we see

$$U^{-1} Q_k U = (\mathrm{e}^{-r\widetilde{A}})_{ki} Q_i, \qquad U^{-1} P_k U = (\mathrm{e}^{-rA})_{ki} P_i. \tag{2}$$

Operating U on the N-mode coordinate eigenstate $|\overrightarrow{q}\rangle = \pi^{-n/4} \exp\left[-\frac{1}{2}\widetilde{\overrightarrow{q}}\overrightarrow{q} + \sqrt{2}\widetilde{\overrightarrow{q}}a^\dagger - \frac{1}{2}\widetilde{a}^\dagger a^\dagger\right]|0\rangle$, (where $\widetilde{a}^\dagger = \left(a_1^\dagger, a_2^\dagger, \ldots, a_n^\dagger\right)$, $\widetilde{\overrightarrow{q}} = (q_1, q_2, \ldots, q_n)$, $\widetilde{A}, a^\dagger$ and $\overrightarrow{q}$ are the transpositions of $\widetilde{a}^\dagger$, $\widetilde{\overrightarrow{q}}$, A, respectively.), i.e.,

$$U|\overrightarrow{q}\rangle = |\Lambda|^{1/2}|\Lambda\overrightarrow{q}\rangle,\ \ \Lambda = \mathrm{e}^{-r\widetilde{A}}, \qquad |\Lambda| \equiv \det\Lambda, \tag{3}$$

and using

$$U = \int \mathrm{d}^n q U|\overrightarrow{q}\rangle\langle\overrightarrow{q}| = |\Lambda|^{1/2}\int \mathrm{d}^n q |\Lambda\overrightarrow{q}\rangle\langle\overrightarrow{q}|, \qquad U^\dagger = U^{-1}, \tag{4}$$

we have

$$U^{-1}Q_k U = |\Lambda| \int \mathrm{d}^n q U|\overrightarrow{q}\rangle\langle\Lambda\overrightarrow{q}|Q_k \int \mathrm{d}^n q' U|\Lambda\overrightarrow{q'}\rangle\langle\overrightarrow{q'}| = (\Lambda Q)_k, \tag{5}$$

which is consistent with equation (2). Thus U can be expressed in the coordinate representation,

$$U = \exp[\mathrm{i}r Q_i A_{ij} P_j] = \sqrt{\det \mathrm{e}^{-r\widetilde{A}}} \int \mathrm{d}^n q |\Lambda\overrightarrow{q}\rangle\langle\overrightarrow{q}|. \tag{6}$$

Using the IWOP technique, we put U into a normal ordering form

$$U = \pi^{-n/2}|\Lambda|^{1/2}\int_{-\infty}^{\infty} \mathrm{d}^n q : \exp\left[-\frac{1}{2}\widetilde{\overrightarrow{q}}(1 + \widetilde{\Lambda}\Lambda)\overrightarrow{q} + \sqrt{2}\widetilde{\overrightarrow{q}}(\widetilde{\Lambda}a^\dagger + a) - \frac{1}{2}(\widetilde{a}a + \widetilde{a}^\dagger a^\dagger) - \widetilde{a}^\dagger a\right]. \tag{7}$$

By the mathematical formula

$$\int_{-\infty}^{\infty} \mathrm{d}^n x \exp[-\widetilde{x}Fx + \widetilde{x}v] = \pi^{n/2}(\det F)^{-1/2} \exp\left[\frac{1}{4}\widetilde{v}F^{-1}v\right], \tag{8}$$

we perform the integration in equation (7) and obtain the explicit normal ordering form of U:

$$U = |\Lambda|^{1/2}|N|^{-1/2} \exp\left[\tfrac{1}{2}\widetilde{a^\dagger}(\Lambda N^{-1}\widetilde{\Lambda} - I)a^\dagger\right] : \exp[\widetilde{a^\dagger}(\Lambda N^{-1} - I)a] : \exp\left[\tfrac{1}{2}\widetilde{a}(N^{-1} - I)a\right], \tag{9}$$

where $N = \frac{1}{2}(I + \widetilde{\Lambda}\Lambda)$, $\widetilde{\Lambda} = \exp(-rA)$, $\Lambda = \exp(-r\widetilde{A})$.

3. New N-mode squeezed vacuum state

Operating U in equation (9) on the N-mode vacuum state leads us to the new N-mode squeezed vacuum state

$$U|\mathbf{0}\rangle = |\Lambda|^{1/2}|N|^{-1/2} \exp\left[\tfrac{1}{2}\widetilde{a^\dagger}(\Lambda N^{-1}\widetilde{\Lambda} - I)a^\dagger\right]|\mathbf{0}\rangle. \tag{10}$$

In this section we proceed to give the explicit expression of $U|\mathbf{0}\rangle$.

When A is an even order, the eigenvalues of $-A$ are

$$\lambda_m = \mathrm{e}^{2m\pi\,\mathrm{i}/n}\ (n = 2p); \qquad m = 1, 2, \ldots, n, \qquad p \in N, \tag{11}$$

and the corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ -\lambda_m \\ \vdots \\ (-\lambda_m)^{n-1} \end{pmatrix}, \qquad m = 1, 2, \ldots, n. \tag{12}$$

Then we have

$$N = \frac{1}{2}(I + \widetilde{\Lambda}\Lambda) = \frac{1}{2n}\begin{pmatrix} h_n & h_1 & h_2 & \cdots & h_{n-1} \\ h_1 & h_n & h_1 & \cdots & h_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_n \end{pmatrix} = \frac{1}{2n}G,$$

$$h_k = g_k,\ h_n = n + g_n,\ g_k = (-1)^k \sum_{m=1}^{n} \cos\frac{2mk\pi}{n}\,\mathrm{e}^{2\mu_m r}, \tag{13}$$

$$\mu_m = \mathbb{R}\lambda_m = \cos\frac{2m\pi}{n},\ g_k = g_{n-k},\ 1 \leqslant k \leqslant n.$$

Let

$$f(\lambda) = h_n + h_1\lambda + h_2\lambda^2 + \cdots + h_{n-1}\lambda^{n-1}, \qquad B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \cdots & \lambda_n^{n-2} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}, \tag{14}$$

the explicit expression of $f(\lambda_j)$ is

$$\begin{aligned} f(\lambda_j) &= n + \frac{1}{2}\sum_{k=1}^{n}\sum_{m=1}^{n}\left[\mathrm{e}^{(1+\frac{2(m+j)}{n})k\pi\,\mathrm{i}} + \mathrm{e}^{(1+\frac{2(j-m)}{n})k\pi\,\mathrm{i}}\right]\mathrm{e}^{2\mu_m r} \\ &= n(1 + \mathrm{e}^{2\mu_{p+j}r}), \qquad 1 \leqslant j \leqslant 2p. \end{aligned} \tag{15}$$

Then equation (13) yields

$$\det N = \left(\frac{1}{2n}\right)^n \det G = \frac{1}{2^n}\prod_{j=1}^{n}(1 + \mathrm{e}^{2\mu_j r}). \tag{16}$$

It is easy to see $GB = BJ$, $J = \mathrm{diag}(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n))$ and $G^{-1} = BJ^{-1}B^{-1}$. Now using equations (14) and (15), we have

$$G^{-1} = \frac{1}{n^2}\begin{pmatrix} \sum_{j=1}^n \frac{1}{1+\mathrm{e}^{2\mu_{p+j}r}} & \sum_{j=1}^n \frac{\lambda_j}{1+\mathrm{e}^{2\mu_{p+j}r}} & \cdots & \sum_{j=1}^n \frac{\lambda_j^{n-1}}{1+\mathrm{e}^{2\mu_{p+j}r}} \\ \sum_{j=1}^n \frac{\lambda_j}{1+\mathrm{e}^{2\mu_{p+j}r}} & \sum_{j=1}^n \frac{1}{1+\mathrm{e}^{2\mu_{p+j}r}} & \cdots & \sum_{j=1}^n \frac{\lambda_j^{n-2}}{1+\mathrm{e}^{2\mu_{p+j}r}} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^n \frac{\lambda_j^{n-1}}{1+\mathrm{e}^{2\mu_{p+j}r}} & \sum_{j=1}^n \frac{\lambda_j^{n-2}}{1+\mathrm{e}^{2\mu_{p+j}r}} & \cdots & \sum_{j=1}^n \frac{1}{1+\mathrm{e}^{2\mu_{p+j}r}} \end{pmatrix}. \tag{17}$$

For $N^{-1} = 2nG^{-1}$ and $\Lambda N^{-1}\widetilde{\Lambda} - I = I - N^{-1}$, then

$$\widetilde{a}^\dagger N^{-1} a^\dagger = \frac{2}{n}\sum_{m,j,k=1}^{n} \frac{\mathbb{R}\left(\lambda_k^{j-m}\right)}{1 + \mathrm{e}^{2\mu_{p+k}r}} a_m^\dagger a_j^\dagger = \frac{2}{n}\sum_{m,j,k=1}^{n} \frac{\left(\cos\frac{2k\pi}{n}\right)^{j-m}}{1 + \mathrm{e}^{2\mu_{p+k}r}} a_m^\dagger a_j^\dagger. \tag{18}$$

It follows that

$$\begin{aligned} U|\mathbf{0}\rangle &= C\exp\left[\frac{1}{2}\widetilde{a}^\dagger(I - N^{-1})a^\dagger\right]|\mathbf{0}\rangle \\ &= C\exp\left(-\frac{1}{2}\sum_{m=1}^{n} a_m^2 + \frac{1}{n}\sum_{m,j,k=1}^{n} \frac{\left(\cos\frac{2k\pi}{n}\right)^{j-m}}{1 + \mathrm{e}^{2\mu_{p+k}r}} a_m^\dagger a_j^\dagger\right)|\mathbf{0}\rangle, \end{aligned} \tag{19}$$

where $C = |\Lambda|^{1/2}|N|^{-1/2} = \left[\frac{1}{2^n}\prod_{j=1}^n(1 + \mathrm{e}^{2\mu_k r})\right]^{-\frac{1}{2}}$.

Especially, when $\mathrm{e}^r \to 0$,

$$U|\mathbf{0}\rangle|_{r\to-\infty} \sim \exp\left[-\frac{1}{2}\sum_{m=1}^{n} a_m^{\dagger 2} + \frac{1}{n}\sum_{m,j,k=1}^{n} (\cos 2k\pi)^{j-m} a_m^\dagger a_j^\dagger\right]|\mathbf{0}\rangle \equiv |\rangle_{se}. \tag{20}$$

Similarly, if n is an odd number, the eigenvalues of $-A$ are

$$\lambda_m = \mathrm{e}^{(1+2m)\pi\,\mathrm{i}/n}, \qquad m = 1, 2, \ldots, n, \qquad n = 2p + 1. \tag{21}$$

Using

$$f(\lambda_j) = n(1 + \mathrm{e}^{2\mu_{p+j}r}), \qquad 1 \leqslant j \leqslant 2p + 1,$$

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -\lambda_1 & -\lambda_2 & \cdots & -\lambda_n \\ \cdots & \cdots & \cdots & \cdots \\ (-\lambda_1)^{n-2} & (-\lambda_2)^{n-2} & \cdots & (-\lambda_n)^{n-2} \\ (-\lambda_1)^{n-1} & (-\lambda_2)^{n-1} & \cdots & (-\lambda_n)^{n-1} \end{pmatrix}, \tag{22}$$

and $G^{-1} = BJ^{-1}B^{-1}$, we have

$$G^{-1} = \frac{1}{n^2}\begin{pmatrix} \sum_{j=1}^n \frac{1}{1+\mathrm{e}^{2\mu_{p+j}r}} & \sum_{j=1}^n \frac{-\lambda_j}{1+\mathrm{e}^{2\mu_{p+j}r}} & \cdots & \sum_{j=1}^n \frac{(-\lambda_j)^{n-1}}{1+\mathrm{e}^{2\mu_{p+j}r}} \\ \sum_{j=1}^n \frac{-\lambda_j}{1+\mathrm{e}^{2\mu_{p+j}r}} & \sum_{j=1}^n \frac{1}{1+\mathrm{e}^{2\mu_{p+j}r}} & \cdots & \sum_{j=1}^n \frac{(-\lambda_j)^{n-2}}{1+\mathrm{e}^{2\mu_{p+j}r}} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^n \frac{(-\lambda_j)^{n-1}}{1+\mathrm{e}^{2\mu_{p+j}r}} & \sum_{j=1}^n \frac{(-\lambda_j)^{n-2}}{1+\mathrm{e}^{2\mu_{p+j}r}} & \cdots & \sum_{j=1}^n \frac{1}{1+\mathrm{e}^{2\mu_{p+j}r}} \end{pmatrix}. \tag{23}$$

It then follows

$$\widetilde{a^{\dagger}}N^{-1}a^{\dagger} = \frac{2}{n}\sum_{m,j,k=1}^{n}(-1)^{j-m}\frac{\mathbb{R}\left(\lambda_k^{j-m}\right)}{1+\mathrm{e}^{2\mu_{p+k}r}}a_m^{\dagger}a_j^{\dagger} = \frac{2}{n}\sum_{m,j,k=1}^{n}\frac{\left[-\cos\frac{(1+2k)\pi}{n})\right]^{j-m}}{1+\mathrm{e}^{2\mu_{p+k}r}}a_m^{\dagger}a_j^{\dagger}, \tag{24}$$

and

$$U|\mathbf{0}\rangle = |\Lambda|^{1/2}|N|^{-1/2}\exp\left\{-\frac{1}{2}\sum_{m=1}^{n}a_m^{\dagger 2}+\frac{1}{n}\sum_{m,j,k=1}^{n}\frac{\left[-\cos\frac{(1+2k)\pi}{n}\right]^{j-m}}{1+\mathrm{e}^{2\mu_{p+k}r}}a_m^{\dagger}a_j^{\dagger}\right\}|\mathbf{0}\rangle, \tag{25}$$

where $C = |\Lambda|^{1/2}|N|^{-1/2} = \left[\frac{1}{2^n}\prod_{j=1}^{n}(1+\mathrm{e}^{2\mu_k r})\right]^{-\frac{1}{2}}$.

Especially, when $\mathrm{e}^r \to 0$,

$$U|0\rangle|_{r\to-\infty} \sim \exp\left\{-\frac{1}{2}\sum_{m=1}^{n}a_m^{\dagger 2}+\frac{1}{n}\sum_{m,j,k=1}^{n}\left[-\cos\frac{(1+2k)\pi}{n}\right]^{j-m}a_m^{\dagger}a_j^{\dagger}\right\}|\mathbf{0}\rangle \equiv |\rangle_{so}. \tag{26}$$

Comparing equations (20) and (26), we obtain

$$U|\mathbf{0}\rangle|_{r\to-\infty} \sim \exp\left[\frac{2}{n}\sum_{k>l=1}^{n}a_l^{\dagger}a_k^{\dagger}-\sum_{j=1}^{n}\frac{n-2}{2n}\left(a_j^{\dagger}\right)^2\right]|\mathbf{0}\rangle \equiv |\rangle_s, \tag{27}$$

where n is an arbitrary integer.

It is interesting to observe that $|\rangle_s$ is just the common eigenvector the N-compatible Jacobian operators in an N-body case with zero eigenvalue [14], i.e.,

$$P|\rangle_s = \sum_{i=1}^{n}p_i|\rangle_s = 0; \qquad \xi_j|\rangle_s = \left\{\left[\sum_{k=j+1}^{n}\mu_k\right]^{-1}\sum_{k=j+1}^{n}\mu_k Q_k - Q_j\right\}|\rangle_s = 0,$$

as $[P,\xi_j] = 0$, $j = 1,2,\ldots,n-1$, $\mu_k = m_k/M$, $M = \sum_{i=1}^{n}m_i$. Since the common eigenvector of N-compatible Jacobian operators is an entangled state, the state $|\rangle_s$ is also an entangled state. This state is experimentally attainable by the use of momentum-squeezed coherence states and position-squeezed coherence states and balanced optical BSs, it tends toward the general perfect EPR-type entangled state in the limit of infinite squeezing. The details will be discussed in section 5.

4. Variances of the *n*-mode quadratures

The quadratures in the N-mode case should be defined as

$$X_1 = \frac{1}{\sqrt{2n}}\sum_{k=1}^{n}Q_k, \qquad X_2 = \frac{1}{\sqrt{2n}}\sum_{k=1}^{n}P_k, \qquad [X_1,X_2] = \frac{\mathrm{i}}{2}. \tag{28}$$

The expectation values of the quadratures in the state $|\rangle_s$ are $\langle X_1\rangle = \langle X_2\rangle = 0$; we see that the corresponding variance is

$$(\Delta X_1)^2 =_s \left\langle\left|x_1^2\right\rangle_s = \frac{1}{4n}\sum_{ji}(\Lambda\widetilde{\Lambda})_{ij} = \frac{1}{4}\,\mathrm{e}^{-2r}, \tag{29}$$

$$(\Delta X_2)^2 =_s \left\langle\left|x_2^2\right|\right\rangle_s = \frac{1}{4n}\sum_{ji}(\Lambda\widetilde{\Lambda})_{ij}^{-1} = \frac{1}{4}\,\mathrm{e}^{2r}, \tag{30}$$

which has the similar standard form to the two-mode case. Equations (29) and (30) clearly indicate that U is the correct N-mode squeezing operator for the N-mode quadratures in equation (28).

5. Optical network for producing the state $|\rangle_s$

The basic operation of optical devices (beam splitters, optical fiber, and phase shifter) based on quantum optics components is the transformation of a set of incoming states into another set by a unitary transformation. Such transformations can be performed by using optical networks. In this section we design such an optical network that the light beams (one mode of zero-position eigenstate $|x = 0\rangle_1$ and the other modes of zero-momentum eigenstates $|p = 0\rangle_2 \otimes |p = 0\rangle_3 \otimes \cdots \otimes |p = 0\rangle_n$) entering the N-input ports of this network will be changed into an N-partite entangled state. That is to say that the network plays the role of transforming N-single-mode-squeezed states ($n - 1$ light fields maximally squeezed in the P direction and one light field in the X direction) incident on the network to the entangled state $|\rangle_s$ in equation (27). In the Fock space $|x = 0\rangle_i$ and $|p = 0\rangle_i$ are expressed as

$$\begin{aligned} |x = 0\rangle_i &\backsim \exp\left(-\frac{1}{2}a_i^{\dagger 2}\right)|0\rangle_i, \\ |p = 0\rangle_i &\backsim \exp\left(\frac{1}{2}a_i^{\dagger 2}\right)|0\rangle_i. \end{aligned} \tag{31}$$

Hence the function of this optical network can be represented by a unitary operator R, and R should meet the following requirement:

$$\begin{aligned} &R|x = 0\rangle_1 \otimes |p = 0\rangle_2 \otimes |p = 0\rangle_3 \otimes \cdots \otimes |p = 0\rangle_n \to |\rangle_s \\ &\quad = \exp\left[\frac{2}{n}\sum_{k>l=1}^{n} a_l^\dagger a_k^\dagger - \sum_{j=1}^{n}\frac{n-2}{2n}(a_j^\dagger)^2\right]|\mathbf{0}\rangle, \end{aligned} \tag{32}$$

which is just equation (27) as indicated. That is to say

$$R\left(a_1^{\dagger 2} - a_2^{\dagger 2} - \cdots - a_n^{\dagger 2}\right)R^{-1} = R\widetilde{a}^\dagger E a^\dagger R^{-1} = \widetilde{a}^\dagger B a^\dagger, \tag{33}$$

where

$$E = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 \end{pmatrix}; \qquad B = \frac{1}{n}\begin{pmatrix} 2-n & 2 & \ldots & 2 \\ 2 & 2-n & \ldots & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \ldots & 2-n \end{pmatrix}.$$

In order to solve R, we suppose

$$R\widetilde{a}^\dagger R^{-1} = \widetilde{a}^\dagger \widetilde{G}, \qquad R a R^{-1} = G_{ij} a_j = a_i'. \tag{34}$$

Then from equation (34), we see that G must satisfy the matrix equation

$$\widetilde{G} E G = B. \tag{35}$$

Its solution is an orthogonal matrix

$$G = \begin{pmatrix} \frac{1}{\sqrt{n}} & \sqrt{\frac{n-1}{n}} & 0 & \ldots & 0 \\ \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n(n-1)}} & \sqrt{\frac{n-2}{n-1}} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{(n-1)(n-2)}} & \ldots & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{36}$$

where $G_{i,1} = \frac{1}{\sqrt{n}}$, $(i = 1, 2, \ldots, n)$; $G_{i,i+1} = \sqrt{\frac{n-i}{n-i+1}}$, $(i = 1, 2, \ldots, n-1)$, $G_{ij} = -\frac{1}{\sqrt{(n-j+1)(n-j+2)}}$, $(i = 2, 3, \ldots, n,\ j = 2, 3, \ldots, i)$, the others $G_{ij} = 0$.

The corresponding R may be constructed by the coherent state representation and the orthogonal matrix G:

$$\begin{aligned} R &= \int \prod_{i=1}^{n} \frac{\mathrm{d}^2 z_i}{\pi} |G_{ij} z i_j\rangle\langle z_i| \\ &= \int \prod_{i=1}^{n} \frac{\mathrm{d}^2 z_i}{\pi} : \exp\left[\sum_{i=1}^{n}\left(-|z_i|^2 + \sum_{j=1}^{n} a_i^\dagger G_{ij} z_j\right) + z_i^* a_i - a_i^\dagger a_i\right] : \\ &=: \exp\left[\widetilde{a_i^\dagger}(G - I)a\right] := \exp\left[\widetilde{a_i^\dagger}(InG)a\right]. \end{aligned} \tag{37}$$

To obtain the optical transfer evolution in equations (33) and (34), we extract an interacting Hamiltonian from the unitary transformation (33) or (34) of quantum states. A systematic prescription for obtaining Hamiltonian for preassigned unitary transformations of quantum states has been proposed in [15, 16]. That is by mapping the classical c-number transformation in a coherent state basis onto quantum-mechanical operators of the Fock space and using the IWOP technique to find the Hamiltonian. Let $InG = \mathrm{i}tK$, with $K^\dagger = K$, then the time-evolution operator is $R(t) = \exp\left\{\mathrm{i}t\widetilde{a_i^\dagger} RKa\right\}$. According to $\mathrm{i}\frac{\partial R(t)}{\partial t} = HR(t)$, we obtain the corresponding Hermitian Hamiltonian

$$H = -\sum_{i,j=1}^{n} a_i^\dagger K_{ij} a_j. \tag{38}$$

Moreover, according to van Loock and Braunstein ’ s method [17], the state $|\rangle_s$ can be generated from n-squeezed modes of the field emitted by the optical parametric optical parametric oscillators (OPOs) below threshold (i.e., optical parametric amplifiers (OPAs)) and appropriately balanced beam splitters. Let

$$\widehat{B}_{ij}(\theta) : \begin{cases} a_i \to a_i \cos\theta + a_j \sin\theta \\ a_j \to a_i \sin\theta - a_j \cos\theta, \end{cases}$$

$$\widehat{N}_{1,\ldots,n} = \widehat{B}_{n-1,n}(\pi/4)\widehat{B}_{n-2,n-1}(\cos^{-1} 1/\sqrt{3}) \cdots \widehat{B}_{12}(\cos^{-1} 1/\sqrt{n}). \tag{39}$$

Applying the beam-splitter operator $\widehat{N}_{1,\ldots,n}$ to a zero-momentum eigenstate in mode 1 and $N-1$ zero-position eigenstates in mode 2 to N, we can obtain the N-mode entangled state $|\rangle_s$. This state is an eigenstate with total momentum zero and relative positions $x_i - x_j = 0\,(i, j = 1, 2, \ldots, n)$.

In summary, we have shown that the operator $U = \exp\left[\mathrm{i}r\left(\sum_{i=1}^{n-1} Q_i P_{i+1} + Q_n P_1\right)\right]$ is an N-mode squeezing operator for the N-mode quadratures exhibiting the standard squeezing; the corresponding squeezed vacuum state in the N-mode Fock space is derived. The entanglement involved in such a state is explained. The optical network for producing the N-mode squeezed state is also constructed. The N-mode squeezed state and entangled state may have potential use in theoretically analyzing N-partite quantum teleportation.

Appendix A

Here we give a rigorous proof for equation (13). Note that $h_n = n + g_n$ and $h_k = g_k$ for $k \neq n$, then we have

$$
\begin{aligned}
f(\lambda_j) &= n + \sum_{k=1}^{n} g_k \lambda_j^k \\
&= n + \sum_{k=1}^{n} \left[(-1)^k \sum_{m=1}^{n} \frac{\lambda_m^k + \overline{\lambda_m}^k}{2} \, \mathrm{e}^{2\mu_m r} \right] \lambda_j^k \\
&= n + \frac{1}{2} \sum_{m=1}^{n} \left[\sum_{k=1}^{n} (-1)^k \left(\lambda_m^k + \overline{\lambda_m}^k \right) \mathrm{e}^{2\mu_m r}. \right.
\end{aligned}
\qquad \text{(A.1)}
$$

For the case $n = 2p$. Since $(-1)^k = \mathrm{e}^{m\pi \,\mathrm{i}/n}$, we see that

$$
f(\lambda_j) = n + \frac{1}{2} \sum_{m=1}^{n} \sum_{k=1}^{n} \left[\mathrm{e}^{1+\frac{2(m+j)}{n})k\pi \,\mathrm{i}} + \mathrm{e}^{1+\frac{2(j-m)}{n})k\pi \,\mathrm{i}} \right] \mathrm{e}^{2\mu_m r}. \qquad \text{(A.2)}
$$

Set $\beta_1 = \mathrm{e}^{1+\frac{2(m+j)}{n})k\pi \,\mathrm{i}}$, $\beta_1 = \mathrm{e}^{1+\frac{2(j-m)}{n})k\pi \,\mathrm{i}}$, we have $\beta_1^n = \beta_2^n = \mathrm{e}^{n\pi \,\mathrm{i}} = 1$. For $1 - \lambda^n = (1-\lambda)(1 + \lambda + \ldots + \lambda^{n-1})$, we have

$$
\sum_{k=1}^{n} \beta_1^k = \sum_{k=0}^{n-1} \beta_1^k = \sum_{k=1}^{n} \mathrm{e}^{1+\frac{2(m+j)}{n})k^2\pi \mathrm{i}} = \begin{cases} 0, & 1 + \frac{2(m+j)}{n} \neq \text{even number} \\ n, & 1 + \frac{2(m+j)}{n} = \text{even number} z. \end{cases} \qquad \text{(A.3)}
$$

As a result of equations (A1)–(A3), we obtain

$$
f(\lambda_j) = \begin{cases} n(1 + \mathrm{e}^{2\mu_{p-j} r}), & 1 \leqslant j < p, \\ n(1 + \mathrm{e}^{2\mu_p r}), & j = p, \\ n(1 + \mathrm{e}^{2\mu_{p+j} r}), & p < j \leqslant p. \end{cases} = n(1 + \mathrm{e}^{2\mu_j r}), \qquad 1 \leqslant j \leqslant 2p \qquad \text{(A.4)}
$$

References

[1] Ekert A and Josza R 1996 *Rev. Mod. Phys.* **68** 733
[2] DiVincenzo D P 1995 *Science* **270** 255
[3] Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 2881
[4] Vaidman L 1994 *Phy. Rev.* A **49** 1473
[5] Furusawa A, Sorensen J L, Braunstein S L, Fuchs C A, Kimble H J and Polzik E 1998 *Science* **28** 223
[6] Braunstein S L and Kimble H J 1998 *Phys. Rev. Lett.* **80** 869
[7] Kuzmich A, Mandel L and Bigelow N P 2000 *Phys. Rev. Lett.* **85** 1594
[8] Duan L-M, Cirac J I, Zoller P and Polzik E S 2000 *Phys. Rev. Lett.* **85** 5643
[9] Feng X-L, Li R-X and Xu Z-Z 2005 *Phys. Rev.* A **71** 032335
[10] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709
[11] Fan H 2001 *Phys. Lett.* A **286** 81
[12] Fan H and Fan Y 1996 *Phys. Rev.* A **54** 958
[13] Fan H and Yu G 2002 *Phys. Rev.* A **65** 033829
[14] Albeverio S, Fei S-M and Song T-q 2005 *Phys. Lett.* A **338** 182
[15] Fan H and Xiao M 1997 *Quantum Semiclass. Opt.* **9** 53
[16] Fan H 1994 *Commun. Theor. Phys.* **21** 509
[17] van Loock P and Braunstein S L 2000 *Phys. Rev. Lett.* **84** 3482